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YEA BIG

17 Jan. 2021

This contest was held on Zoom, on Monday, 17 January 2022, 1PM-4PM.

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> Misplaced problem!
>
> Unnamed AoPS spammer

§-1 Credits

Youth EUCLID would like to thank the following individuals for their important contributions the contest:

- **Problem authors:** Aaron Chen, Catherine Li, Neal Yan, Alex Bai, Tiger Zhang, Dennis Chen
- **Testsolvers:** Ethan Han, Neal Yan, Aditya Chandrasekhar, Harry Zhang, Gore Fee

Special thanks to Aaron for providing the logistical backbone of the contest.

§0 Problems

1. Find the value of $6\sqrt[3]{(2^9+2^1-24)(2^9-2^2+192)}$.

2. A four-digit integer $ABCD$ (with $A \neq 0$) is said to be **balanced** if $A+B=C+D$. Find the number of balanced four-digit integers.

3. On a whiteboard in an empty classroom, there is a 3-digit number $\underline{abc}$ written down in base x: $\underline{abc}_x$. The first student, Albert, comes in, and correctly converts this number to $\underline{633}_{10}$. Albert then decides to erase his answer and the base of the original problem, and replaces it with $x+1$ before leaving. Next, Bob comes into the room and correctly converts the new number $\underline{abc}_{x+1}$ to $\underline{750}_{10}$. Like Albert, he decides to erase his answer, but instead replaces the base of the original problem with $x+2$. Finally, Charlie comes in the room. He correctly converts the existing number $\underline{abc}_{x+2}$ to $\underline{877}_{10}$. Find $\underline{abc}_{10}-x$.

   (The notation $\underline{a_1\dots a_n}_m$ denotes the number $\underline{a_1\dots a_n}$ base m, as opposed to $a_1\cdot a_2\cdots a_n$.)

4. In quadrilateral $ABCD$, $\angle A=\angle B=\angle C=75^\circ$, and $BC=2$. The infimal and supremal lengths of AB are $\sqrt{a}-\sqrt{b}$ and $\sqrt{c}+\sqrt{d}$ respectively, for positive integers a,b,c,d. Find $a+b+c+d$.
   (The **infimum** and **supremum** of a variable quantity are its greatest lower and least upper bounds, respectively.)

5. For each positive integer n, let the roots of quadratic equation $x^2+(2n+1)x+n^2=0$ be r_n and s_n. Given that
$$\sum_{n=3}^{20}\frac{1}{(1+r_n)(1+s_n)}=a/b$$
   for some relatively prime positive integers a,b, find the remainder when $a+b$ is divided by 1000.

6. Determine the number of noncongruent right triangles with integral sides $a,b,c<1000$ satisfying the following conditions:

   - One of the legs is even, and it is 1 less than the hypotenuse;
   - $\gcd(a,b,c)=1$.

7. A number is said to be **mountainous** if its digits strictly increase to a peak then strictly decrease, and all digits are non-zero. For instance, 12321 is a mountainous number, but 12331 and 12435 are not. If k is the number of five-digit mountainous numbers there are, find the remainder when k is divided by 1000.

   Clarification: The peak cannot occur at the first or last digit. For instance 12579 and 85432 are not regarded as mountainous.

8. Find the number of distinct integers $1\le n\le 1000$ expressible as $x^3+y^3+z^3-3xyz$ for some positive integers x,y,z.

9. Benjamin makes 10 different cards, on each of which he writes two different numbers from $\{1,2,3,4,5\}$. He distributes these cards into 5 plates numbered 1 through 5, such that a card can only be put into a correspondingly numbered plate—for example, the card with the numbers 3 and 5 written on it can be put into either plate 3 or plate 5. Benjamin considers an arrangement **good** if plate 1 contains more cards than any of the other four plates. Find the number of good arrangements.

10. In triangle ABC, the bisector of $\angle BAC$ meets side BC at point D, with $BD/DC=2/3$. If $\angle ADB=60^\circ$, then $(AB+AC)/BC=\sqrt{a}$ for some positive integer a. Find a.

11. Find the number of ordered n-tuples of positive integers $(k_1,k_2,\dots,k_n)$ such that
$$\sum_{m=1}^{n}k_m=5n-4,\text{ and}$$
$$\sum_{m=1}^{n}\frac{1}{k_m}=1.$$

12. Alice and Bob are playing a game with 4 stacks of coins with a, b, c, d coins, respectively, with no three piles equal. The players take turns moving alternately, with Alice going first. On each move, the player chooses two non-empty stacks. Suppose they (currently) have x and y coins, in either order. Then, the player removes $\min(x, y)$ coins from both stacks. The player unable to move loses. If $a = 300$ and Bob has a winning strategy, find the minimum value of $b + c + d$.

13. In triangle ABC, with $\angle A = 50^\circ$, let point D be the midpoint of $\overline{BC}$. Points E, F are on $\overline{AC}, \overline{AB}$ respectively, such that $CE = BF = BC/2$. Let I be the incenter of $\triangle ABC$, and $(BFI)^*$ and (CEI) intersect at a point $P \neq I$. If $AE - AF = 216$, find PD.

14. Let $\mathcal{S}$ be the set of all integers x such that $0 < x < 167$ and there exists an integer k such that

$$k^2 \equiv x + 2 \pmod{167}.$$

Let N be the product of all elements of $\mathcal{S}$. What is the remainder when N is divided by 167?

15. Let acute $\triangle ABC$ have orthocenter H. Let D, E, F be the feet of the altitudes from A, B, C respectively. Let $BD = 3/2$, $CD = 11/2$, and $AH = 17/\sqrt{15}$. Point P lies on $\overline{EF}$ such that $\overline{PA} \parallel \overline{BC}$. If the tangent from P to (AEF) not parallel to $\overline{BC}$ meets the median from A at K, and $\overline{HK}$ meets line EF at X, then BX is expressible as a/b for relatively prime positive integers a, b. Compute $a + b$.

$^*(A_1 \ldots A_n)$ denotes the circumcircle of cyclic polygon $(A_1 \ldots A_n)$.

§1 AIME 2022/1, proposed by Aaron Chen

Find the value of $6\sqrt[3]{(2^9+2^1-24)(2^9-2^2+192)}$.

Ignore the 6 for now.

$$\sqrt[3]{(2^9+2^1-24)(2^9-2^2+192)}=\sqrt[3]{(514-24)(512+188)}=\sqrt[3]{490\cdot 700}=70.$$

Multiplying by 6 gives us $\boxed{420}$. Ha ha ha.

§2 AIME 2022/2, proposed by Catherine Li

A four-digit integer $ABCD$ (with $A\neq 0$) is said to be **balanced** if $A+B=C+D$. Find the number of balanced four-digit integers.

Sum on the common sum mentioned in the problem, which we call N:

- $N\in[1,9]$; N options for first two digits, $N+1$ options for last two. For this case we get $1\cdot 2+\cdots+9\cdot 10=2\binom{11}{3}=330$;
- $N\in[10,18]$; $N'=19-N$ options for each pair of digits mentioned in the problem. This case gives us $9^2+\cdots+1^2=9\cdot 10\cdot 19/6=285$;

Total count is $\boxed{615}$, the end.

§3 AIME 2022/3, proposed by Aaron Chen

On a whiteboard in an empty classroom, there is a 3-digit number $\underline{abc}$ written down in base x: $\underline{abc}_x$. The first student, Albert, comes in, and correctly converts this number to $\underline{633}_{10}$. Albert then decides to erase his answer and the base of the original problem, and replaces it with $x+1$ before leaving. Next, Bob comes into the room and correctly converts the new number $\underline{abc}_{x+1}$ to $\underline{750}_{10}$. Like Albert, he decides to erase his answer, but instead replaces the base of the original problem with $x+2$. Finally, Charlie comes in the room. He correctly converts the existing number $\underline{abc}_{x+2}$ to $\underline{877}_{10}$. Find $\underline{abc}_{10} - x$.

(The notation $\underline{a_1 \ldots a_n}_m$ denotes the number $\underline{a_1 \ldots a_n}$ base m, as opposed to $a_1 \cdot a_2 \cdots a_n$.)

Mathematically worded:

$$\begin{aligned} \underline{abc}_x &= 633; \\ \underline{abc}_{x+1} &= 750; \\ \underline{abc}_{x+2} &= 877. \end{aligned}$$

(Righthand side numbers are base 10.) Express these as quadratics relating to x:

$$ax^2 + bx + c = 633; \tag{3.1}$$

$$a(x+1)^2 + b(x+1) + c = 750; \tag{3.2}$$

$$a(x+2)^2 + b(x+2) + c = 877. \tag{3.3}$$

(3.2) − (3.1) and ((3.3) − (3.1))/2 give us

$$a(2x+1) + b = 117 \text{ and} \tag{3.4}$$

$$a(2x+2) + b = 122. \tag{3.5}$$

Now (3.5) − (3.4) gives $a = 5$;

$$\Rightarrow 5(2x+1) + b = 117 \Rightarrow 10x + b = 112.$$

$$b < x \Rightarrow b = 112 \mod 10 = 2, x = (112 - b)/10 = 11.$$

Finally $c = 633 - x(ax + b) = 633 - 11(55 + 2) = 6$. Requested difference is then $526 - 11 = \boxed{515}$.

§4 AIME 2022/4, proposed by Catherine Li

In quadrilateral $ABCD$, $\angle A = \angle B = \angle C = 75°$, and $BC = 2$. The infimal and supremal lengths of AB are $\sqrt{a} - \sqrt{b}$ and $\sqrt{c} + \sqrt{d}$, respectively, for positive integers a, b, c, d. Find $a + b + c + d$.
(The **infimum** and **supremum** of a variable quantity are its greatest lower and least upper bounds, respectively.)

Let lines BA and CD meet at point E. Fix B, C while varying A, D (linearly) along segments BE and CE respectively. Clearly each moves linearly with respect to the other. A diagram quickly shows us that the infimum and supremum are represented by two limiting cases, namely $D \to B$ and $A \to E$, respectively.
Observe that the sides of a 30°-75°-75° triangle are in the ratio 1 : 1 :. If we let $r = (\sqrt{6} + \sqrt{2})/2$, then requested bounds are $2/r$ and $2r$, which turn out to be

$$\sqrt{6} - \sqrt{2} \text{ and } \sqrt{6} + \sqrt{2}$$

respectively.
The desired sum is $6 + 2 + 6 + 2 = \boxed{016}$.

§5 AIME 2022/5, proposed by Aaron Chen

For each positive integer n, let the roots of quadratic equation $x^2 + (2n + 1)x + n^2 = 0$ be r_n and s_n. Given that

$$\sum_{n=3}^{20} \frac{1}{(1 + r_n)(1 + s_n)} = a/b$$

for some relatively prime positive integers a, b, find $a + b$.

By Vieta,

$$(1 + r_n)(1 + s_n) = 1 + (r_n + s_n) + (r_n s_n) = 1 - (2n + 1) + n^2 = n(n - 2).$$

The series then telescopes:

$$\sum_{n=3}^{20} \frac{1}{n(n-2)} = \frac{1}{2} \sum_{n-3}^{20} \left(\frac{1}{n-2} - \frac{1}{n} \right) = \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} - \frac{1}{19} - \frac{1}{20} \right) = 531/760.$$

The requested sum is congruent to $\boxed{291} \pmod{1000}$.

§6 AIME 2022/6, proposed by Aaron Chen

Determine the number of noncongruent right triangles with integral sides $a, b, c < 1000$ satisfying the following conditions:

- One of the legs is even, and it is 1 less than the hypotenuse;
- $\gcd(a, b, c) = 1$.

Let the mentioned sides be of lengths $2n$, $2n+1$. Then the other leg is $\sqrt{4n + 1}$, which is supposed to be integral. We may ignore the relative primality condition because we already have $\gcd(2n, 2n + 1) = \gcd(2n, 1) = 1$.

Let $4n + 1$ equal the square $(2k + 1)^2$, so that $n = k(k + 1)$. This leads to Pythag triples of the form

$$(2k + 1, 2k(k + 1), 2k(k + 1) + 1).$$

For the first condition in the problem it is sufficient that the hypotenuse be less than 1000:

$$2k(k + 1) + 1 < 1000 \Rightarrow k(k + 1) \leq 499 \Rightarrow k \leq 21.$$

As we may verify that all positive integer k work (modulo the < 1000 condition), the answer is the number of distinct integers in $[1, 21]$, which is easily seen to be $\boxed{021}$.

§7 AIME 2022/7, proposed by Dennis Chen

A number is said to be **mountainous** if its digits strictly increase to a peak then strictly decrease, and all digits are non-zero. For instance, 12321 is a mountainous number, but 12331 and 12435 are not. If k is the number of distinct five-digit mountainous numbers, find the remainder when k is divided by 1000.

We do casework on the location of the peak digit:

- **Case 1:** Second or fourth digit is largest; in this case, the peak digit is some $k \in [4, 9]$. For the first digit any integer less than k works there are $n - 1$ options, while for the other four we simply pick any 3 distinct integers from $[1, k - 1]$. This case (and its symmetric variant) each have a count of

$$\sum_{k=4}^{9}(k-1)\binom{k-1}{3} = 4\sum_{k=4}^{9}\binom{k}{4} - \sum_{k=4}^{9}\binom{k-1}{3} = 4\binom{10}{5} - \binom{9}{4} = 882;$$

  This subcase yields a subtotal of $2 \cdot 882 = (1764)$;

- **Case 2:** Third digit largest; similarly, peak is an integer $k \in [3, 9]$. For each of the first two and last two digits, we may choose any two distinct integers from $[1, k - 1]$, for a subtotal of

$$\begin{aligned}\sum_{k=3}^{9}\binom{k-1}{2}^2 = \sum_{k=1}^{7}k^2(k+1)^2/4 &= 6\sum_{k=1}^{7}\binom{k}{4} + 12\sum_{k=1}^{7}\binom{k}{3} + 7\sum_{k=1}^{7}\binom{k}{2} + \sum_{k=1}^{7}\binom{k}{1} \\ &= 6\binom{8}{5} + 12\binom{8}{4} + 7\binom{8}{3} + \binom{8}{2} = (1596);\end{aligned}$$

We obtain a total of $1764 + 1596 = 3360 \equiv \boxed{360} \pmod{1000}$.

> *Note.* In this problem, choosing distinct integers disregarded order, because mountainity determines the order in which the digits are assigned to the chosen integers.

§8 AIME 2022/8, proposed by Aaron Chen

Find the number of integers $1 \le n \le 1000$ expressible as $x^3 + y^3 + z^3 - 3xyz$ for some positive integers x, y, z.

Factor the expression as $(x + y + z)\left[\sum_{\text{cyc}}(x - y)^2\right]/2$, and call this $P(x, y, z)$. (The sum is cyclic.) We say that:

Claim — All $n \not\equiv 0 \pmod 3$ exceeding 2 and $n \equiv 0 \pmod 9$ exceeding 9 work, and all other n fail.

Proof. We first provide the following constructions for the cases we claim work:

- $1 \pmod 3$: for all $k \ge 1$ consider $(x, y, z) = (k, k, k + 1)$, giving us $P = (3k + 1)(0^2 + 1^2 + (-1)^2)/2 = 3k + 1$;
- $2 \pmod 3$ is almost identical: consider $(k + 1, k + 1, k)$. (It is probably easier for the contestant to discover this case via $(k, k, k - 1)$.)
- $0 \pmod 9$: $(k, k + 1, k + 2) \Rightarrow (3k + 3)(1^2 + 1^2 + (-2)^2)/2 = 9(k + 1), \forall k \ge 1$.

It remains to show that $v_3(n) = 1$ fails, which we do by showing that if n works and $3 \mid n$, then $9 \mid n$. Indeed, we have $x^3 + y^3 + z^3 - 3xyz \equiv \sum_{\text{cyc}} x^3 = x + y + z$ by Fermat's little theorem, whence $3 \mid (x + y + z)$. Hence 9 divides $(x + y + z)^3 - 3(x + y + z)(xy + yz + zx) = x^3 + y^3 + z^3 - 3xyz$, so we win.

Now for the edge cases $n = 1, 2, 9$ excluded from the constructions:

- **n = 1, 2:** $n = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) \ge (1 + 1 + 1)(1) = 3$, so $n = 1, 2$ fail;
- **n = 9:** $(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 9$, while $x + y + z \ge 1 + 1 + 1 = 3$. Because $x + y + z \mid 9$, $x + y + z = 3$ or 9.
  If $x + y + z = 3$, then $\sum_{\text{cyc}}(x^2 - xy) = 0$, which is bad;
  If $x + y + z = 9$, then
  $$1 = x^2 + y^2 + z^2 - xy - yz - zx = (x + y + z)^2 - 3(xy + yz + zx) = 81 - 3(xy + yz + zx),$$
  implying the absurd
  $$xy + yz + zx = 80/3.$$

It is evident that all three edge cases above fail. □

Having discovered the olympiad-style answer, the counting is routine:

- **Multiples of 9:** $\lfloor 1000/9 \rfloor - 1 = (110)$;
- **Non-multiples of 3:** $1000 - \lfloor 1000/3 \rfloor - 2 = (665)$;

Total count is thus $\boxed{775}$.

§9 AIME 2022/9, proposed by Catherine Li

Benjamin makes 10 different cards, on each of which he writes two different numbers from $\{1, 2, 3, 4, 5\}$. He distributes these cards into 5 plates numbered 1 through 5, such that a card can only be put into a correspondingly numbered plate–for example, the card with the numbers 3 and 5 written on it can be put into either plate 3 or plate 5. Benjamin considers an arrangement **good** if plate 1 contains more cards than any of the other four plates. Find the number of good arrangements.

Solution: We denote the card with a and b written on it by the ordered pair (a, b), where $1 \leq a < b \leq 5$. There are 10 cards and 5 plates, so a good arrangement must have a plate 1 containing at least 3 cards. There are 4 cards that can be put into plate 1, so we have two cases:

- **Case 1:** Plate 1 contains 4 cards. Then the other 6 cards can be put in either of their two valid assignments without violating the goodness condition, giving us $2^6 = 64$ good arrangements;
- **Case 2:** Plate 1 contains 3 cards. There are 4 ways to choose 3 of the 4 cards to put in plate 1. Without loss of generality, put $(1, 2)$, $(1, 3)$, and $(1, 4)$ into plate 1, so that $(1, 5)$ must go to plate 5. Then we have two subcases, noting that plate n has 1 to 2 cards for $2 \leq n \leq 5$:
  - **Subcase 2a:** Plate 5 contains only 1 card. Then $(n, 5)$ must go to plate n for $n = 2, 3, 4$. There are three remaining cards, and $(2, 3)$, $(3, 4)$, $(4, 2)$ must be placed in different plates, so 2 arrangements for this subcase;
  - **Subcase 2b:** Plate 5 contains 2 cards. WLOG it contains $(1, 5)$, $(2, 5)$, so that $(n, 5)$ goes to plate n for $n = 3, 4$. However, for the remaining 3 cards, there are 4 bad arrangements in which one of plates 3, 4 contains 3 cards: $(2, 4)$, $(3, 4)$ to plate 4, $(2, 3)$ to either 2 or 3, or $(2, 3)$, $(3, 4)$ to plate 3, $(2, 4)$ to plate 2 or 4. This subcase yields $3(2^3 - 4) = 12$ arrangements;

Final count is $64 + 4(2 + 12) = \boxed{120}$.

§10 AIME 2022/10, proposed by Catherine Li

In triangle ABC, the bisector of $\angle BAC$ meets side BC at point D, with $BD/DC = 2/3$. If $\angle ADB = 60°$, then $(AB+AC)/BC = \sqrt{a}$ for some positive integer a. Find a.

By the angle bisector theorem we may let $AB, AC = 2x, 3x$, so the the requested number is x^2.
We then compute AD in terms of x:

$$AD = \sqrt{AB \cdot AC \left(1 - BC^2/(AB + AC)^2\right)} = \sqrt{(2x)(3x)\left(1 - 5^2/(5x)^2\right)} = \sqrt{6(x^2 - 1)}.$$

The law of cosines and a bit of algebra finishes the job:

$$AB^2 = AD^2 + BD^2 - AD \cdot BD \Rightarrow 4x^2 = 6(x^2 - 1) + 2^2 - 2\sqrt{6(x^2 - 1)};$$

$$\Rightarrow x^2 - 1 = \sqrt{6(x^2 - 1)}, \sqrt{x^2 - 1} = \sqrt{6}, x^2 = \boxed{007}.$$

§11 AIME 2022/11, proposed by Aaron Chen

Find the number of ordered n-tuples of integers $(k_1, k_2, \ldots, k_n)$ such that

$$\sum_{m=1}^{n} k_m = 5n - 4, \text{ and}$$
$$\sum_{m=1}^{n} \frac{1}{k_m} = 1.$$

Multiplying the two given inequalities and using Cauchy gives

$$5n - 4 = \left(\sum_{m=1}^{n} k_m\right)\left(\sum_{m=1}^{n} \frac{1}{k_m}\right) \geq n^2 \qquad (11.1)$$

whence $n \leq 4$. Now we do some routine casework:

- **n = 1:** $k_1 = 1/k_1 = 1$, so this case gives 1 tuple;
- **n = 2:** $k_1 + k_2 = 6$, $k_1k_2 = k_1 + k_2 = 6$, so that k_1, k_2 are zeros of $x^2 - 6x + 6$. As this quadratic has no integral zeros, this case is impossible;
- **n = 3:** $k_1 + k_2 + k_3 = 11$, $1/k_1 + 1/k_2 + 1 + k_3 = 1$; WLOG let $k_1 \leq k_2 \leq k_3$, so that $k_1 = 2$ or 3. A simple enumeration of all such triples with sum 11 looks like so: $(k_1, k_2, k_3) = (2, 2, 7), (2, 3, 6), (2, 4, 5), (3, 3, 5), (3, 4, 4)$. In this list only $(2, 3, 6)$ works. There are a total of $3! = 6$ tuples in this case;
- **n = 4:** Because equality holds in (11.1), we must have $k_1 = k_2 = k_3 = k_4$ by the equality condition of Cauchy. We easily obtain $k_1 = k_2 = k_3 = k_4 = 4$, yielding 1 tuple.

Total count is $1 + 6 + 1 = \boxed{008}$.

§12 AIME 2022/12, proposed by Tiger Zhang

Alice and Bob are playing a game with 4 stacks of coins with a, b, c, d coins, respectively, with no three piles equal. The players take turns moving alternately, with Alice going first. On each move, the player chooses two non-empty stacks. Suppose they (currently) have x and y coins, in either order. Then, the player removes $\min(x, y)$ coins from both stacks. The player unable to move loses. If $a = 300$ and Bob has a winning strategy, find the minimum value of $b + c + d$.

We can see that each move empties out one pile at a time, unless the two piles have the same number of coins, in which case both piles empty out. It follows that in order for Bob to win, the only necessary condition is that no matter what move Alice makes in her first turn, there must exist two piles with the same number of coins after her move.

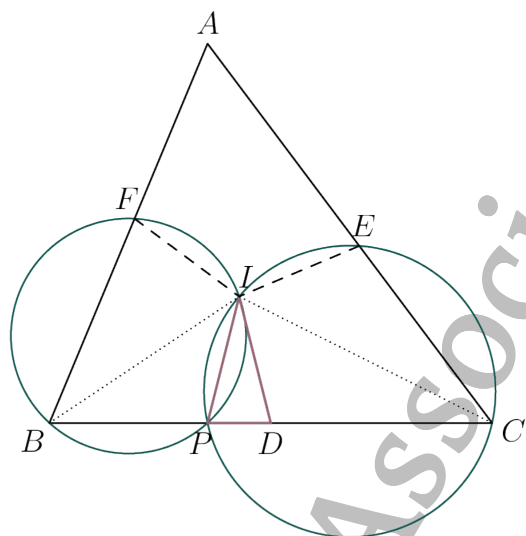
We are given that the first pile has 300; WLOG let the second pile have x. To minimize $b + c + d$, it is clear that we should have $x \leq 300$. Then, we can see that to guarantee the existence of a pair of piles with the same number of coins after Alice chooses the stacks of 300 and x, $300 - x$ is also the coin amount of one of the (other) piles. It is now apparent that to minimize x, we should have $x \leq 150$. Now we will do casework, taking the minimum of the two values of $b + c + d$ obtained:

- **x = 150:** If equality holds, we may see that the necessary combination of b, c, d is $b = c = d = 150$, leading to a sum of 450;
- **x < 150:** Applying a similar argument to the stacks of $300 - x$ and x, we can see that $300 - 2x$ should also be a pile, and $300 - 3x$, and so on. However, we are given that there are only 4 piles, so we must have $300 - 3x = 0$, whence $x = 100$. This leads to $(b, c, d) = (100, 200, 100)$, or a sum of (400).

Since $400 < 450$, we obtain a minimal value of $\boxed{400}$.

§13 AIME 2022/13, proposed by Aaron Chen

In triangle ABC, with $\angle A = 50^\circ$, let point D be the midpoint of $\overline{BC}$. Points E, F are on $\overline{AC}, \overline{AB}$ respectively, such that $CE = BF = BC/2$. Let I be the incenter of $\triangle ABC$, and $(BFI)^\dagger$ and (CEI) intersect at a point $P \neq I$. If $AE - AF = 216$, find PD.



A good diagram leads to the following result:

Claim — $P \in \overline{BC}$.

Proof. We will use some direct angles $\pmod{180^\circ}$, and such angles will be denoted using the symbol $\measuredangle$, as opposed to $\angle$.

It is easy to see that because $\angle DBI = \angle FBI$ and $DB = FB$, D and F are reflections over $\overline{BI}$, whence $\measuredangle IDB = -\measuredangle IFB$. Similarly $\measuredangle IDC = -\measuredangle IEC$.
Using cyclic quadrilaterals $IPBF$, $IPCE$ and the above equalities, the result follows by angle chasing:

$$\measuredangle IPB = \measuredangle IFB = -\measuredangle IDB = -\measuredangle IDC = \measuredangle IEC = \measuredangle IPC,$$

whence B, P, C collinear. □

Next, for the length computation, we utilise:

Claim — $IP = ID$.

Proof. Sufficient to showing $\measuredangle IDB = -\measuredangle IPB$, which will imply that $\triangle IPD$ is isosceles. This also follows by angle chasing:

$$\measuredangle IPB = \measuredangle IFB = -\measuredangle IDB$$

as claimed. □

Let T be the incircle touch point on $\overline{BC}$. From the previous claim we know that T is the midpoint of $\overline{DP}$ by symmetry about $\overline{IT}$.
Let $a, b, c = BC, CA, AB$ and $s = (a+b+c)/2$. Then we are given $|b-c| = 216$, so

$$DP = 2DT = 2\,|BD - BT| = 2\,|a/2 - (s-b)| = 2\,|a/2 - (a+c-b)/2| = |b-c| = \boxed{216}.$$

Observe that $\angle A = 50^\circ$ is a red herring!

$^\dagger(A_1 \ldots A_n)$ denotes the circumcircle of cyclic polygon $(A_1 \ldots A_n)$.

§14 AIME 2022/14, proposed by Alex Bai

Let $\mathcal{S}$ be the set of all integers x such that $0 < x < 167$ and there exists an integer k such that

$$k^2 \equiv x + 2 \pmod{167}.$$

Let N be the product of all elements of $\mathcal{S}$. What is the remainder when N is divided by 167?

Let $\mathcal{T}$ be the set $\{1, \ldots, 83\} \setminus \{13\}$. Then we are asked for

$$\prod_{n \in \mathcal{T}} (n^2 - 2) \mod 167.$$

Note that $n^2 - 2 \equiv n^2 - 169 = (n+13)(n-13) \equiv -(13-n)(13+n)$, and rewrite the product as

$$\prod_{n \in \mathcal{T}} [-(13-n)(13+n)] \mod 167.$$

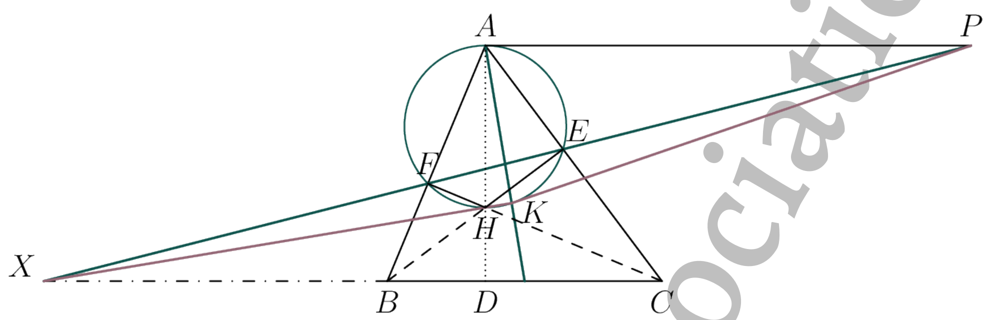
Observe that all reduced residues other than $13 - 13 = 0$ and $13 + 13 = 26$ are included in the product, while 13 is included twice. We may then evaluate the product as follows, noting that $|T| = 83$, an odd number:

$$(-1)^{83} \prod_{n \in \mathcal{T}} 13(166!)/26 \equiv -13(-1)/26 \equiv 1/2 \equiv \boxed{084} \pmod{167}.$$

(In the last line we used Wilson's theorem to get $166! \equiv -1 \pmod{167}$.)

§15 AIME 2022/15, proposed by Neal Yan

Let acute $\triangle ABC$ have orthocenter H. Let D, E, F be the feet of the altitudes from A, B, C respectively, with $BD = 3/2$, $CD = 11/2$, and $AH = 17/\sqrt{15}$. Point P lies on $\overline{EF}$ such that $PA \parallel BC$. If the tangent from P to (AEF) not parallel to $\overline{BC}$ meets the median from A at K, and $\overline{HK}$ meets line EF at X then BX is expressible as a/b for relatively prime positive integers a, b. Compute $a + b$.



The line through A parallel to $\overline{BC}$ is tangent to (AEF), because its diameter $\overline{AH}$ is perpendicular to said line. Let the harmonic conjugate of A wrt $\overline{EF}$ be K'. To show that $K = K'$, we need two claims:

Claim — $\overline{PK'}$ touches (AEF).

Proof. It is well-known that because $(AK'; EF) = -1$, $\overline{EF}$ and the tangents to (AEF) at A, K' concur at a point, which we are given is P. Hence $\overline{PK'}$ touches (AEF) as needed. □

Claim — $\overline{AK'}$ is a median of $\triangle ABC$.

Proof. Because $(AK'; EF) = -1$, $\overline{AK'}$ is a symmedian of $\triangle AEF$. Hence it is a median of $\triangle ABC$ because $\overline{BC}, \overline{EF}$ are antiparallel wrt[‡] $\angle A$. □

Having shown that $K = K'$, the problem is finished via:

Claim — $\overline{HK}, \overline{BC}, \overline{EF}$ are concurrent.

(Equivalently, $X \in \overline{EF}$.)

Proof. We will show that $\overline{EF} \cap \overline{BC} = \overline{HK} \cap \overline{BC}$.

Observe that by a well-known harmonic bundle lemma, $(\overline{EF} \cap \overline{BC}, D; B, C) = -1$, while

$$(\overline{HK} \cap \overline{BC}, D; B, C) \stackrel{H}{=} (KA; EF) = -1.$$

As both $\overline{EF} \cap \overline{BC}$ and $\overline{HK} \cap \overline{BC}$ are the harmonic conjugate of D wrt $\overline{BC}$, they are the same point. □

Now it is routine to compute BX seeing as $(XD; BC) = -1$ and all sides are given: X is on ray CB because $BD < CD$ implies $BX < CX$, and BX may be computed as follows:

$$BX/CX = BD/CD = 3/11, BX/BC = 3/8;$$

$$\Rightarrow BX = \frac{3}{8}BC = 21/8 \Rightarrow \boxed{029}.$$

‡"with respect to"

Remark. Here, the original triangle intended was a 6-7-8 one, but anyways AH is also a red herring.