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YEA BIG

17 Jan. 2021

This contest is being/was held on Zoom, 17 January 2022. Have fun!

§0 Problems

1. Find the value of $6\sqrt[3]{(2^9 + 2^1 - 24)(2^9 - 2^2 + 192)}$.

2. A four-digit integer $ABCD$ (with $A \neq 0$) is said to be **balanced** if $A + B = C + D$. Find the number of balanced four-digit integers.

3. On a whiteboard in an empty classroom, there is a 3-digit number $\underline{abc}$ written down in base x: $\underline{abc}_x$. The first student, Albert, comes in, and correctly converts this number to $\underline{633}_{10}$. Albert then decides to erase his answer and the base of the original problem, and replaces it with $x + 1$ before leaving. Next, Bob comes into the room and correctly converts the new number $\underline{abc}_{x+1}$ to $\underline{750}_{10}$. Like Albert, he decides to erase his answer, but instead replaces the base of the original problem with $x+2$. Finally, Charlie comes in the room. He correctly converts the existing number $\underline{abc}_{x+2}$ to $\underline{877}_{10}$. Find $\underline{abc}_{10} - x$.

   (The notation $\underline{a_1 \dots a_n}_m$ denotes the number $\underline{a_1 \dots a_n}$ base m, as opposed to $a_1 \cdot a_2 \cdots a_n$.)

4. In quadrilateral $ABCD$, $\angle A = \angle B = \angle C = 75°$, and $BC = 2$. The infimal and supremal lengths of AB are $\sqrt{a} - \sqrt{b}$ and $\sqrt{c} + \sqrt{d}$ respectively, for positive integers a, b, c, d. Find $a + b + c + d$.
   (The **infimum** and **supremum** of a variable quantity are its greatest lower and least upper bounds, respectively.)

5. For each positive integer n, let the roots of quadratic equation $x^2 + (2n + 1)x + n^2 = 0$ be r_n and s_n. Given that
$$\sum_{n=3}^{20} \frac{1}{(1 + r_n)(1 + s_n)} = a/b$$
   for some relatively prime positive integers a, b, find the remainder when $a + b$ is divided by 1000.

6. Determine the number of noncongruent right triangles with integral sides $a, b, c < 1000$ satisfying the following conditions:
   - One of the legs is even, and it is 1 less than the hypotenuse;
   - $\gcd(a, b, c) = 1$.

7. A number is said to be **mountainous** if its digits strictly increase to a peak then strictly decrease, and all digits are non-zero. For instance, 12321 is a mountainous number, but 12331 and 12435 are not. If k is the number of distinct five-digit mountainous numbers, find the remainder when k is divided by 1000.

   Clarification: The peak cannot occur at the first or last digit. For instance 12579 and 85432 are not regarded as mountainous.

8. Find the number of distinct integers $1 \leq n \leq 1000$ expressible as $x^3 + y^3 + z^3 - 3xyz$ for some positive integers x, y, z.

9. Benjamin makes 10 different cards, on each of which he writes two different numbers from $\{1, 2, 3, 4, 5\}$. He distributes these cards into 5 plates numbered 1 through 5, such that a card can only be put into a correspondingly numbered plate–for example, the card with the numbers 3 and 5 written on it can be put into either plate 3 or plate 5. Benjamin considers an arrangement **good** if plate 1 contains more cards than any of the other four plates. Find the number of good arrangements.

10. In triangle ABC, the bisector of $\angle BAC$ meets side BC at point D, with $BD/DC = 2/3$. If $\angle ADB = 60°$, then $(AB + AC)/BC = \sqrt{a}$ for some positive integer a. Find a.

11. Find the number of ordered n-tuples of integers $(k_1, k_2, \dots, k_n)$ such that
$$\sum_{m=1}^{n} k_m = 5n - 4, \text{ and}$$
$$\sum_{m=1}^{n} \frac{1}{k_m} = 1.$$

12. Alice and Bob are playing a game with 4 stacks of coins with a, b, c, d coins, respectively, with no three piles equal. The players take turns moving alternately, with Alice going first. On each move, the player chooses two non-empty stacks. Suppose they (currently) have x and y coins, in either order. Then, the player removes $\min(x, y)$ coins from both stacks. The player unable to move loses. If $a = 300$ and Bob has a winning strategy, find the minimum value of $b + c + d$.

13. In triangle ABC, with $\angle A = 50^\circ$, let point D be the midpoint of $\overline{BC}$. Points E, F are $\overline{AC}$, $\overline{AB}$ respectively, so that $CE = BF = BC/2$. Let I be the incenter of $\triangle ABC$, and $(BFI)^*$ and (CEI) intersect at a point $P \neq I$. If $AE - AF = 216$, find PD.

14. Let $\mathcal{S}$ be the set of all integers x such that $0 < x < 167$ and there exists an integer k such that $k^2 \equiv x + 2 \pmod{167}$. Let N be the product of all elements of $\mathcal{S}$. What is the remainder when N is divided by 167?

15. Let acute $\triangle ABC$ have orthocenter H. Let D, E, F be the feet of the altitudes from A, B, C respectively. Let $BD = 3/2$, $CD = 11/2$, and $AH = 17/\sqrt{15}$. Point P lies on $\overline{EF}$ such that $\overline{PA} \parallel \overline{BC}$. If the tangent from P to (AEF) not parallel to $\overline{BC}$ meets the median from A at K, and $\overline{HK}$ meets line EF at X, then BX is expressible as a/b for relatively prime positive integers a, b. Compute $a + b$.

This sentence is false.

$^*(A_1 \dots A_n)$ denotes the circumcircle of cyclic polygon $(A_1 \dots A_n)$.