Addition Is Made Easily

Youth EUCLID

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> “ ‘Good artists borrow, great artists steal.’ ”
>
> Evan Chang

Special thanks to Adam and Selena Ge for being the only testsolvers :p

> *Note.* To make the problem statements less cumbersome, if you get a fractional answer a/b for some of the problems, submit as answer the sum of the numerator and denominator in lowest terms– $a+b$ when $\gcd(a, b) = 1$.

0 Problems

Problem 1. A point P is randomly selected outside circle ω such that the two tangent segments to ω from P form an angle of greater than 60 degrees. What is the probability that they form an angle greater than 90 degrees?

Problem 2. Let m be the smallest positive integer greater than 1000 that is divisible by 13 and its digits sum to 13. Find the last 3 digits of m.

Problem 3. A frog is set at $(0, 0, 0)$ in 3 dimensional space. Each day, it hops 1 unit in exactly one of the three cardinal directions[*]. The probability the frog will hop in a given direction is not dependent on the day or current location (it may be more likely to choose a direction over another). Given that the probability it reaches one of $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ but NOT $(1, 1, 1)$ is $\frac{2}{5}$, what is the probability the frog reaches one of $(2, 0, 0)$, $(0, 2, 0)$, $(0, 0, 2)$ but NOT one of $(3, 0, 0)$, $(0, 3, 0)$, $(0, 0, 3)$?

Problem 4. In regular hexagon $ABCDEF$, define $T = \overline{BF} \cap \overline{CD}$ and $S = \overline{BC} \cap \overline{ET}$. Given that the area of triangle CTS is 140, find the remainder when the floor of the area of $ABCDEF$ is divided by 1000. (Here, $\overline{AB} \cap \overline{CD}$ is shorthand for the intersection of lines AB, CD.)

Problem 5. Aaron, Catherine, and Evan each come up with a "random" sequence of four fair coin flips every minute, recording the results. However, to make the sequences (of four flips each) appear random, they doctor the data by redoing a group of 4 flips when:

- all four flips in the sequence have the same outcome, or
- it begins with the same outcome as the last two recorded groups of 4 flips.

[*] i.e. positive x-, y-, and z-directions

(The offending roll sequence is discarded, and they immediately redo the last group of 4 flips.) As they continue this indefinitely, what proportion of the time do they get different results for the groups of 4 flips (from each other)?

(For example, if they obtain HHTH, HTHH, and TTHT, then they're said to obtain different results, but not for HHTH, TTHT, HHTH. Sequences must exactly match for us to say they got the same result.)

Problem 6. In cyclic quadrilateral $ABCD$ with $AB \cdot CD = AD \cdot BC$, let point P be on $\overline{BD}$ that the perpendicular line to $\overline{AC}$ from P is concurrent with the bisectors of $\angle B, \angle D$ at some point T. If $TB = 5$, $TD = 4$, and $TP = 11$, what is BD?

Problem 7. As x ranges over the positive reals, what is the remainder when the number of values

$$\lfloor x \rfloor + \lfloor x^2 \rfloor + \lfloor x^3 \rfloor + \lfloor x^4 \rfloor + \lfloor x^5 \rfloor$$

attains between 1 and 111111 (inclusive) is divided by 1000?

Problem 8. Let M be the number of ways to re-arrange the letters of MISSISSIPPI such that all the S's are consecutive and all the P's are consecutive. Let N be the number of ways to re-arrange the letters of MISSISSIPPI such that no two $S's$ or $P's$ are consecutive. What is M/N?

Problem 9. In (convex) cyclic quadrilateral $ABCD$ with circumcenter O and diagonals $AC, BD = \sqrt{78}, 13$ respectively, we have $BC = CD$. Let the circumcenter P of $\triangle OAC$ lie on $\overline{BD}$. If the perpendicular from P to $\overline{AC}$ meets the circumcircle of $\triangle OBD$ at a point X on the opposite side of $\overline{AC}$ as P, then $BX/DX = (a - \sqrt{b})/c$ for some positive integers a, b, c with $\gcd(a, b, c) = 1$. Find $a + b + c$.

Problem 10. Sheldon is stuck in a network of rooms numbered 1 to 40, with an exit in the last room. In each other room, there are no windows or doors, but in room k (for all $1 \leq k \leq 40$ (sic)) there is a portal that sends him to room $k - 1$ and $k + 1$ with constant but unequal probabilities as well as travel times (the same for each room). If he is at room 1, the portal necessarily sends him to room 2. Suppose that he immediately enters the portal in a room upon arriving, and that:

- At room 20, the probability he'll exit the network without ever reaching room 10 is 1/2;
- The expected time for him to travel from room 11 to room 21 is 38 seconds;
- However, given that he reaches room 1 along the way, the expected time becomes 48 seconds;

The expected time for Sheldon to travel from room 10 to room 20 is $a - \sqrt{b}$ for integers a, b. What is $a + b$?