Addition Is Made Easily

Youth EUCLID

January 16, 2023

> " 'Good artists borrow, great artists steal.' "
>
> Evan Chang

Special thanks to Adam and Selena Ge for being the only testsolvers :p

> *Note.* To make the problem statements less cumbersome, if you get a fractional answer a/b for some of the problems, submit as answer the sum of the numerator and denominator in lowest terms– $a + b$ when $\gcd(a, b) = 1$.

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0 Problems

Problem 1. A point P is randomly selected outside circle ω such that the two tangent segments to ω from P form an angle of greater than 60 degrees. What is the probability that they form an angle greater than 90 degrees?

Problem 2. Let m be the smallest positive integer greater than 1000 that is divisible by 13 and its digits sum to 13. Find the last 3 digits of m.

Problem 3. A frog is set at $(0, 0, 0)$ in 3 dimensional space. Each day, it hops 1 unit in exactly one of the three cardinal directions*. The probability the frog will hop in a given direction is not dependent on the day or current location (it may be more likely to choose a direction over another). Given that the probability it reaches one of $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ but NOT $(1, 1, 1)$ is $\frac{2}{5}$, what is the probability the frog reaches one of $(2, 0, 0)$, $(0, 2, 0)$, $(0, 0, 2)$ but NOT one of $(3, 0, 0)$, $(0, 3, 0)$, $(0, 0, 3)$?

Problem 4. In regular hexagon $ABCDEF$, define $T = \overline{BF} \cap \overline{CD}$ and $S = \overline{BC} \cap \overline{ET}$. Given that the area of triangle CTS is 140, find the remainder when the floor of the area of $ABCDEF$ is divided by 1000. (Here, $\overline{AB} \cap \overline{CD}$ is shorthand for the intersection of lines AB, CD.)

Problem 5. Aaron, Catherine, and Evan each come up with a "random" sequence of four fair coin flips every minute, recording the results. However, to make the sequences (of four flips each) appear random, they doctor the data by redoing a group of 4 flips when:

- all four flips in the sequence have the same outcome, or
- it begins with the same outcome as the last two recorded groups of 4 flips.

(The offending roll sequence is discarded, and they immediately redo the last group of 4 flips.) As they continue this indefinitely, what proportion of the time do they get different results for the groups of 4 flips (from each other)?

(For example, if they obtain HHTH, HTHH, and TTHT, then they're said to obtain different results, but not for HHTH, TTHT, HHTH. Sequences must exactly match for us to say they got the same result.)

Problem 6. In cyclic quadrilateral $ABCD$ with $AB \cdot CD = AD \cdot BC$, let point P be on $\overline{BD}$ that the perpendicular line to $\overline{AC}$ from P is concurrent with the bisectors of $\angle B, \angle D$ at some point T. If $TB = 5$, $TD = 4$, and $TP = 11$, what is BD?

Problem 7. As x ranges over the positive reals, what is the remainder when the number of values

$$\lfloor x \rfloor + \lfloor x^2 \rfloor + \lfloor x^3 \rfloor + \lfloor x^4 \rfloor + \lfloor x^5 \rfloor$$

attains between 1 and 111111 (inclusive) is divided by 1000?

Problem 8. Let M be the number of ways to re-arrange the letters of MISSISSIPPI such that all the S's are consecutive and all the P's are consecutive. Let N be the number of ways to re-arrange the letters of MISSISSIPPI such that no two $S's$ or $P's$ are consecutive. What is M/N?

Problem 9. In (convex) cyclic quadrilateral $ABCD$ with circumcenter O and diagonals $AC, BD = \sqrt{78}, 13$ respectively, we have $BC = CD$. Let the circumcenter P of $\triangle OAC$ lie on $\overline{BD}$. If the perpendicular from P to $\overline{AC}$ meets the circumcircle of $\triangle OBD$ at a point X on the opposite side of $\overline{AC}$ as P, then $BX/DX = (a - \sqrt{b})/c$ for some positive integers a, b, c with $\gcd(a, b, c) = 1$. Find $a + b + c$.

Problem 10. Sheldon is stuck in a network of rooms numbered 1 to 40, with an exit in the last room. In each other room, there are no windows or doors, but in room k (for all $1 \le k \le 40$ (sic)) there is a portal that sends him

*i.e. positive x-, y-, and z-directions to room $k-1$ and $k+1$ with constant but unequal probabilities as well as travel times (the same for each room). If he is at room 1, the portal necessarily sends him to room 2. Suppose that he immediately enters the portal in a room upon arriving, and that:

- At room 20, the probability he'll exit the network without ever reaching room 10 is $1/2$;
- The expected time for him to travel from room 11 to room 21 is 38 seconds;
- However, given that he reaches room 1 along the way, the expected time becomes 48 seconds;

The expected time for Sheldon to travel from room 10 to room 20 is $a-\sqrt{b}$ for integers a, b. What is $a+b$?

1 Solutions

1.1 AIME 2023/1 (Sheldon Tan)

A point P is randomly selected outside circle ω such that the two tangent segments to ω from P form an angle of greater than 60 degrees. What is the probability that they form an angle greater than 90 degrees?

Let O be the center of the circle. Then we want $2 < OP^2 < 4$. The desired area is an annulus as shown, giving an answer of

$$\frac{\pi(\sqrt{2}^2 - 1^2)}{\pi(2^2 - 1^2)} = \frac{1}{3} \Rightarrow \boxed{004}.$$

1.2 AIME 2023/2 (Aaron Chen)

Let m be the smallest positive integer greater than 1000 that is divisible by 13 and its digits sum to 13. Find the last 3 digits of m.

The central observation of the problem is that $m \equiv 13 \pmod{117}$, which follows since its digit sum means $m \equiv 13 \pmod 9$ while we're given $m \equiv 0 \pmod{13}$. Luckily, the first integer $\equiv 13 \pmod{117}$ exceeding 1000 works: $1\boxed{066}$.

1.3 AIME 2023/3 (Evan Chang)

A frog is set at $(0,0,0)$ in 3 dimensional space. Each day, it hops 1 unit in exactly one of the three cardinal directions[†]. The probability the frog will hop in a given direction is not dependent on the day or current location (it may be more likely to choose a direction over another). Given that the probability it reaches one of $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ but NOT $(1,1,1)$ is $\frac{2}{5}$, what is the probability the frog reaches one of $(2,0,0)$, $(0,2,0)$, $(0,0,2)$ but NOT one of $(3,0,0)$, $(0,3,0)$, $(0,0,3)$?

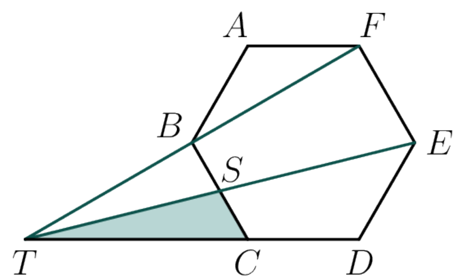
Let a, b, c be the probabilities the frog moves in each direction. The given probability becomes $\sum_{\text{cyc}} 2ab(a+b) = 2/5$, while the desired is

$$\sum_{\text{cyc}} a^2(1-a) = \frac{1}{2}\sum_{\text{cyc}} 2ab(a+b) = \frac{1}{5} \Rightarrow \boxed{006}.$$

[†] i.e. positive x-, y-, and z-directions

1.4 AIME 2023/4 (Catherine Li)

In regular hexagon $ABCDEF$, define $T = \overline{BF} \cap \overline{CD}$ and $S = \overline{BC} \cap \overline{ET}$. Given that the area of triangle CTS is 140, find the area of $ABCDEF$. (Here, $\overline{AB} \cap \overline{CD}$ is shorthand for the intersection of lines AB, CD.)



By coordinates, say, T is the reflection of F in $\overline{BC}$. Then we may observe that $BECT$ is a parallelogram, so S is the midpoint of $\overline{BC}$, $\overline{TE}$. As a result, we obtain

$$[ABCDEF] = 6[BCD] = 3[BCE] = 3[BTC] = 6[CTS] = \boxed{840}.$$

1.5 AIME 2023/5 (Sheldon Tan)

Aaron, Catherine, and Evan each come up with a "random" sequence of four fair coin flips every minute, recording the results. However, to make the sequences (of four flips each) appear random, they doctor the data by redoing a group of 4 flips when:

- all four flips in the sequence have the same outcome, or
- it begins with the same outcome as the last two recorded groups of 4 flips.

(The offending roll sequence is discarded, and they immediately redo the last group of 4 flips.) As they continue this indefinitely, what proportion of the time do they get different results for the groups of 4 flips (from each other)?

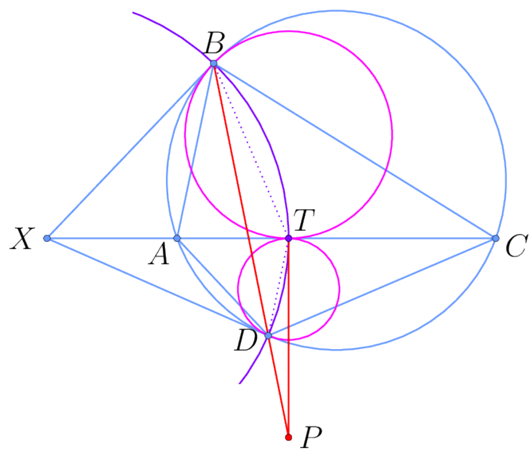
(For example, if they obtain HHTH, HTHH, and TTHT, then they're said to obtain different results, but not for HHTH, TTHT, HHTH. Sequences must exactly match for us to say they got the same result.)

This is a "troll" problem. The probability that any two given people have the same move sequence at any given time is just 1 divided by the total number of valid sequences, which is $2^4 - 2 = 14$. So the requested answer is just

$$\frac{13}{14}\frac{12}{14} \Rightarrow 39 + 49 = \boxed{088}.$$

1.6 AIME 2023/6 (Neal Yan)

In cyclic quadrilateral $ABCD$ with $AB \cdot CD = AD \cdot BC$, let point P be on $\overline{BD}$ that the perpendicular line to $\overline{AC}$ from P is concurrent with the bisectors of $\angle B$, $\angle D$ at some point T. If $TB = 5$, $TD = 4$, and $TP = 11$, what is BD?



By angle bisector theorem we know that $T \in \overline{AC}$.

Claim – $\overline{TP}$ is tangent to (BDT).

Proof. Draw the circles ω_b, ω_d tangent to $\overline{AC}$ at T and $(ABCD)$ at B, D respectively. Their radical center $X = \overline{BB} \cap \overline{DD} \cap \overline{AC}$ is also the circumcircle of $\triangle BDT$. As $\overline{XAC} \perp \overline{TP}$, the claim follows. □

Since $\frac{PB}{PT} = \frac{5}{4}$ and $\frac{PD}{PT} = \frac{4}{5}$, we have

$$BD = \frac{9}{20} PT = \frac{99}{20} \Rightarrow \boxed{119}.$$

1.7 AIME 2023/7 (Sheldon Tan)

As x ranges over the positive reals, what is the remainder when the number of values

$$\lfloor x \rfloor + \lfloor x^2 \rfloor + \lfloor x^3 \rfloor + \lfloor x^4 \rfloor + \lfloor x^5 \rfloor$$

attains between 1 and 111111 (inclusive) is divided by 1000?

The main idea is that as we continuously increase x, the given function increments in the following cases at some given value of x:

- **Increments of 1;** most values, ignore;
- **Increments of 2;** this occurs when x^2, x^4 are integers but not x (and thus not x^3 either); as x^2 can be anything from 1 to 100 inclusive, other than a square, we get $10^2 - 10 = (90)$ values "skipped" by these $+ = 2$'s;
- **Increments of 5;** this occurs when x itself is an integer. Values skipped: $4 \cdot 10 = (40)$;

The requested answer is $111111 - 40 - 90 = 110\boxed{981}$.

1.8 AIME 2023/8 (Aaron Chen)

Let M be the number of ways to re-arrange the letters of MISSISSIPPI such that all the S's are consecutive and all the P's are consecutive. Let N be the number of ways to re-arrange the letters of MISSISSIPPI such that no two $S's$ or $P's$ are consecutive. What is M/N?

We can see that M is just the number of arrangements of MSPIIII, of which there are $7!/4! = 210$ ways.

To find N, consider all distributions of the four S's in 11 slots (the total number of letters) such that no two of them are in consecutive slots.

- By stars and bars, this has $\binom{7+2-1}{5-1} = (70)$ solutions;
- Among the remaining letters, there is one M, two P's, and four I's, so ignoring the condition for the two letters P, there are $5 \cdot \binom{7}{2} = (105)$ ways to distribute the rest of the letters.

This results in $70 \cdot 105 = 7350$ possibilities. Now, we have to subtract the number of arrangements with the two P's consecutive. We can treat these as one letter P, for a total of 10 slots instead of 11.

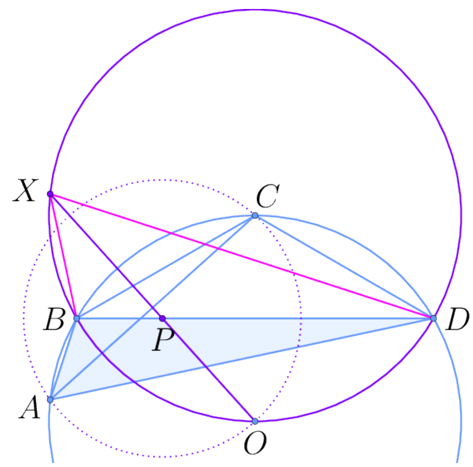
- For the distributions of the four S's, by stars and bars, we obtain $\binom{6+2-1}{5-1} = (35)$ ways;
- For the other letters we get $6 \cdot 5 = (30)$;

As we have to subtract $35 \cdot 30$ from the subtotal 7350, $N = 7350 - 1050 = 6300$.

As a result $M/N = 210/6300 = \frac{1}{30} \implies \boxed{031}$.

1.9 AIME 2023/9 (Neal Yan)

In (convex) cyclic quadrilateral $ABCD$ with circumcenter O and diagonals $AC, BD = \sqrt{78}, 13$ respectively, we have $BC = CD$. Let the circumcenter P of $\triangle OAC$ lie on $\overline{BD}$. If the perpendicular from P to $\overline{AC}$ meets the circumcircle of $\triangle OBD$ at a point X on the opposite side of $\overline{AC}$ as P, then $BX/DX = (a - \sqrt{b})/c$ for some positive integers a, b, c with $\gcd(a, b, c) = 1$. Find $a + b + c$.



For brevity let ℓ be the perpendicular bisector of $\overline{AC}$ aka the perpendicular from P to $\overline{AC}$.

Claim – $\angle BOD = 120^\circ$.

Proof. Observe that P lies on $\overline{BD}$ and the perpendicular bisector of $\overline{OC}$ which are supposed to be parallel. Thus the two mentioned lines are coincident which implies the result. □

Lemma 1 – In triangle ABC with $\angle A = 60^\circ$, circumcenter O, and orthocenter H, $AH = AO$.

Proof. Omitted. □

Claim 2 – X is the orthocenter of $\triangle ABD$.

Proof. Proceed by phantom points, letting H be the desired orthocenter; then

$$\measuredangle BHD = 60^\circ = \measuredangle BOD \implies H \in (OBD).$$

Meanwhile (by design) C, A are the circumcenter and orthocenter of $\triangle HBD$ so lemma 1 means $HA = HC$ or $H \in \ell$. By design $H = X$. □

Lemma 2 – In a triangle with sides a, b, c, circumradius R, circumcenter O and orthocenter H,

$$OH^2 = 9R^2 - (a^2 + b^2 + c^2).$$

Proof. Also omitted. □

Now we extract the answer; in some awful notation, let $BH = u$, $DH = v$. Combine lemma 2 on $\triangle HBD$ with $BD^2 = u^2 - uv + v^2$ (law of cosines) to obtain $AC^2 = (u - v)^2$ and thus the system

$$\begin{aligned} (u - v)^2 &= AC^2 = 78; \\ u^2 - uv + v^2 &= BD^2 = 169; \end{aligned}$$

Homogenizing yields

$$13(u - v)^2 = 6(u^2 - uv + v^2) \Rightarrow 7u^2 - 20uv + 7v^2 = 0 \Rightarrow \frac{u}{v} = \frac{10 \pm \sqrt{51}}{7} \Rightarrow \boxed{068}.$$

1.10 AIME 2023/10 (Sheldon Tan)

Sheldon is stuck in a network of rooms numbered 1 to 40, with an exit in the last room. In each other room, there are no windows or doors, but in room k (for all $1 \le k \le 40$ (sic)) there is a portal that sends him to room $k-1$ and $k+1$ with constant but unequal probabilities as well as travel times (the same for each room). If he is at room 1, the portal necessarily sends him to room 2. Suppose that he immediately enters the portal in a room upon arriving, and that:

- At room 20, the probability he'll exit the network without ever reaching room 10 is 1/2;
- The expected time for him to travel from room 11 to room 21 is 38 seconds;
- However, given that he reaches room 1 along the way, the expected time becomes 48 seconds;

The expected time for Sheldon to travel from room 10 to room 20 is $a - \sqrt{b}$ for integers a, b. What is $a + b$?

Let $p, q(= 1-p)$ be the respective probabilities Sheldon is transported forwards (+1) / backwards (−1) in any room other than the first.

> **Claim –** p is given by
> $$\left(\frac{q}{p}\right)^{10} = \frac{\sqrt{5}-1}{2}.$$

Proof. This is provable from the first condition alone. Let p_k denote the probability Sheldon reaches room 40 without reaching room 10 en route (for any $10 \le k \le 40$). Then, $p_{10} = 0, p_{40} = 1$, and $\forall 10 < k < 40$,

$$p_k = qp_{k-1} + pp_{k+1} \iff p_{k+1} - p_k = \frac{q}{p}(p_k - p_{k-1}).$$

Now recall that we're given $p_{20} - p_{10} = p_{20} = \frac{1}{2}$. By telescoping,

$$\frac{1}{2} = p_{20} - p_{10} = (p_{20} - p_{19}) + \cdots + (p_{11} - p_{10}) = d\left(1 + \left(\frac{q}{p}\right)^1 + \cdots + \left(\frac{q}{p}\right)^9\right) = d\frac{1 - \left(\frac{q}{p}\right)^{10}}{1 - \left(\frac{q}{p}\right)}$$

and similarly

$$1 = p_{40} - p_{10} = d\frac{1 - \left(\frac{q}{p}\right)^{30}}{1 - \left(\frac{q}{p}\right)}$$

where $d = p_{11} - p_{10}$. Dividing these gives an easily solvable quadratic in $\left(\frac{q}{p}\right)^{10}$:

$$\left(\frac{q}{p}\right)^{20} + \left(\frac{q}{p}\right)^{10} + 1 = 2.$$ □

We now turn to the last two items. For brevity, let E denote the event where Sheldon never reaches 1 while travelling from 11 to 21, which is guaranteed eventually.

By similar logic as the above proof, we obtain

$$P(E) = \frac{1 - \left(\frac{q}{p}\right)^{10}}{1 - \left(\frac{q}{p}\right)^{20}} = \frac{1}{1 + \left(\frac{1-p}{p}\right)^{10}} \overset{\text{claim}}{=} \frac{\sqrt{5}-1}{2}.$$

Let F denote the expected time to complete E (i.e. arrive at room 21 while avoiding room 1). Then

$$P(E) \cdot F + (1 - P(E)) \cdot 48 = 38 \Rightarrow F = 43 - 5\sqrt{5}.$$

Finally, observe that in all routes from 11 to 21 avoiding 1, when Sheldon reaches 2, he *must* be sent to room 3 next. Shifting labels by -1[‡], F is also the expected time for Sheldon to travel from room 10 to room 20 (unrestricted), so we obtain an answer of $43 + 125 = \boxed{168}$.

Remark. one of the problems of all time

Warning. This problem was intended as a practical joke lmao

[‡]consider "weight" of each route, then note that set of probabilities wrt position is still the same, but with room 2 acting as "room 1"