Proposal Compilation– Version Public

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> *Remark.* Version Private… you may have seen some of those.
>
> Public proposals aren't that great because they're from 2021, when I didn't have that many math ideas, sorry…

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0 Problems

Problem 1 (OMMC Sample 2023/2/3). In acute triangle ABC, equilateral triangle ABX and regular hexagon $BCPQRS$ are externally constructed on their respective sides. Let line XP intersect $\overline{AB}$, $\overline{BC}$, and $\overline{AR}$ at Y, Z, K, respectively. Prove that $\overline{AZ}$, $\overline{BK}$, $\overline{CY}$ are concurrent.

Problem 2. Let

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{2k}{k} (n-k+1) 2^{-n-3k}.$$

What is S^2?

> *Remark.* Unused for negative quality reasons.

Problem 3 (YEMO 3). Variable triangles ABC and DEF share a fixed incircle ω and circumcircle Ω. Let ω_a be the A-mixtilinear incircle in $\triangle ABC$, and similarly for ω_d. Determine (as the triangles vary) the locus of the intersection of the common external tangents to these two circles.

Problem 4 (Mock AIME 2023/6). [*] In cyclic quadrilateral $ABCD$ with $AB \cdot CD = AD \cdot BC$, let point P be on $\overline{BD}$ that the perpendicular line to $\overline{AC}$ from P is concurrent with the bisectors of $\angle B, \angle D$ at some point T. If $TB = 5$, $TD = 4$, and $TP = 11$, what is BD?

Problem 5 (Mock AIME 2023/9). In (convex) cyclic quadrilateral $ABCD$ with circumcenter O and diagonals $AC, BD = \sqrt{78}, 13$ respectively, we have $BC = CD$. Let the circumcenter P of $\triangle OAC$ lie on $\overline{BD}$. If the perpendicular from P to $\overline{AC}$ meets the circumcircle of $\triangle OBD$ at a point X on the opposite side of $\overline{AC}$ as P, then $BX/DX = (a-\sqrt{b})/c$ for some positive integers a, b, c with $\gcd(a, b, c) = 1$. Find $a+b+c$.

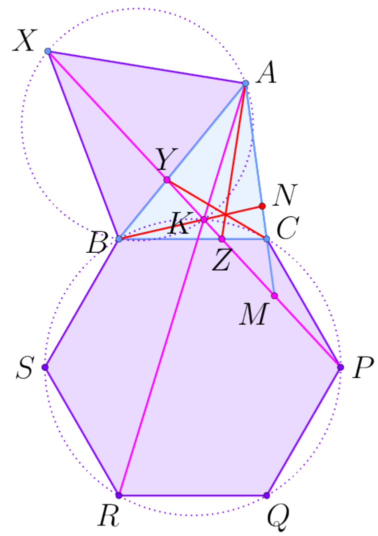
[*] `https://unity858.github.io/yea/`

1 Solutions

1.1 External regular polygons (OMMC Sample 2023/2/3)

`https://www.ommcofficial.org/sample`

In acute triangle ABC, equilateral triangle ABX and regular hexagon $BCPQRS$ are externally constructed on their respective sides. Let line XP intersect $\overline{AB}, \overline{BC}$, and $\overline{AR}$ at Y, Z, K, respectively. Prove that $\overline{AZ}, \overline{BK}, \overline{CY}$ are concurrent.



Extend lines XP, BK to meet line AC at points M, N, respectively. Then, we have,

Claim – $(AC; MN) = -1$.

Proof. First, because rotations are spiral similarities, and $\overline{AR}$, $\overline{XP}$ are related by a $\pi/3$ one, K is also the second intersection of (ABX) and (BPR) distinct from B, that is, the second intersection of the circumcircles of the two regular polygons.

Due to this we have $\angle AKC = 2\pi - \angle AKB - \angle BKC = \pi/2$.

Now we obtain $\angle NKC = \pi - \angle BKC = \pi/6$, and $\angle MKC = \angle PKC = \pi/6$, and $\overline{CK}$ bisects $\angle MKN$. By a well-known lemma the claim is proven. □

Now Ceva-Menelaus in reverse finishes the problem.

Remark. This problem was made in May '21, actually…

1.2 Elaborate 69 joke (unused)

Let

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{2k}{k}(n-k+1)2^{-n-3k}.$$

What is S^2?

The sum can be rewritten as

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left((-1)^k\binom{2k}{k}2^{-4k}\right)\left(2^{-(n-k)}\binom{n-k+2}{2}\right).$$

Seeing the convolution, we do a second rewrite:

$$S=\left(\sum_{a=0}^{\infty}(-1)^k\binom{2k}{k}2^{-4k}\right)\left(\sum_{b=0}^{\infty}2^{-b}\binom{b+2}{2}\right).$$

The first sum above evaluates as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}2^{-4n}&=\sum_{n=0}^{\infty}(-1/4)^n\binom{2n}{n}(1/4)^n\\&=\sum_{n=0}^{\infty}\binom{-1/2}{n}(1/4)^n\\&=(1+1/4)^{-1/2}=(2/\sqrt{5});\end{aligned}$$

(The last line follows by the binomial theorem.)

The second sum is easier:

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{n+2}{2}2^{-n}&=\sum_{n=0}^{\infty}\binom{-2}{n}(-1/2)^n\\&=(1-1/2)^{-2}=(4);\end{aligned}$$

Multiplying these sums together gives

$$S=(2/\sqrt{5})(4)=8/\sqrt{5}\Rightarrow S^2=\boxed{64/5}.$$

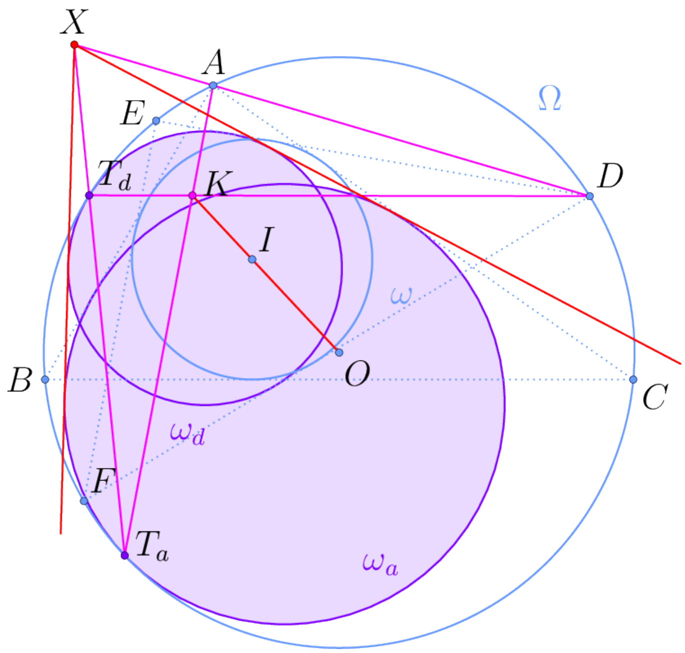
Remark. We use generalized binomial coefficients. Some alg-manip yields $\binom{-1/2}{n}=(-1/4)^n\binom{2n}{n}$ as used in the computation.

Remark. Made in December 2021.

1.3 Monge spam be like... (YEMO 3)

https://unity858.github.io/yea/yemo-2022.html

Variable triangles ABC and DEF share a fixed incircle ω and circumcircle Ω. Let ω_a be the A-mixtilinear incircle in $\triangle ABC$, and similarly for ω_d. Determine (as the triangles vary) the locus of the intersection of the common external tangents to these two circles.

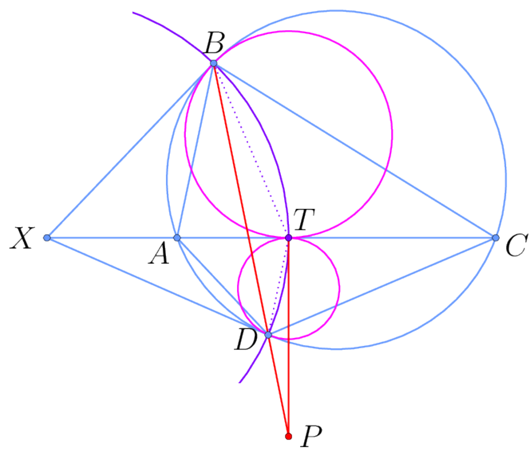


Let the mixtilinears touch Ω at T_a, T_d, and let K, X denotes the exsimilicenters of (Ω, ω) (fixed) and (ω_a, ω_d), the desired. Applying Monge to all possible triplets out of the four circles implies that $K = \overline{AT_a} \cap \overline{DT_D}$ while $X = \overline{AD} \cap \overline{T_aT_d}$. By Brokard, it follows that X lies on the polar of K wrt Ω, a **fixed line**.

1.4 Harmonic quad amerigeo (Mock AIME 2023/6)

`https://unity858.github.io/yea/`

In cyclic quadrilateral $ABCD$ with $AB \cdot CD = AD \cdot BC$, let point P be on $\overline{BD}$ that the perpendicular line to $\overline{AC}$ from P is concurrent with the bisectors of $\angle B, \angle D$ at some point T. If $TB = 5$, $TD = 4$, and $TP = 11$, what is BD?



By angle bisector theorem we know that $T \in \overline{AC}$.

Claim – $\overline{TP}$ is tangent to (BDT).

Proof. Draw the circles ω_b, ω_d tangent to $\overline{AC}$ at T and $(ABCD)$ at B, D respectively. Their radical center $X = \overline{BB} \cap \overline{DD} \cap \overline{AC}$ is also the circumcircle of $\triangle BDT$. As $\overline{XAC} \perp \overline{TP}$, the claim follows. □

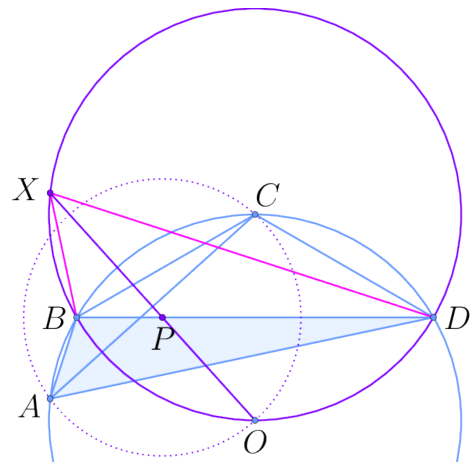
Since $\frac{PB}{PT} = \frac{5}{4}$ and $\frac{PD}{PT} = \frac{4}{5}$, we have

$$BD = \frac{9}{20} PT = \boxed{\frac{99}{20}}.$$

1.5 60-deg anti-prob (Mock AIME 2023/9)

`https://unity858.github.io/yea/`

In (convex) cyclic quadrilateral $ABCD$ with circumcenter O and diagonals $AC, BD = \sqrt{78}, 13$ respectively, we have $BC = CD$. Let the circumcenter P of $\triangle OAC$ lie on $\overline{BD}$. If the perpendicular from P to $\overline{AC}$ meets the circumcircle of $\triangle OBD$ at a point X on the opposite side of $\overline{AC}$ as P, then $BX/DX = (a - \sqrt{b})/c$ for some positive integers a, b, c with $\gcd(a, b, c) = 1$. Find $a + b + c$.



For brevity let ℓ be the perpendicular bisector of $\overline{AC}$ aka the perpendicular from P to $\overline{AC}$.

Claim – $\measuredangle BOD = 120°$.

Proof. Observe that P lies on $\overline{BD}$ and the perpendicular bisector of $\overline{OC}$ which are supposed to be parallel. Thus the two mentioned lines are coincident which implies the result. □

Lemma 1 – In triangle ABC with $\angle A = 60°$, circumcenter O, and orthocenter H, $AH = AO$.

Proof. Omitted. □

Claim 2 – X is the orthocenter of $\triangle ABD$.

Proof. Proceed by phantom points, letting H be the desired orthocenter; then

$$\measuredangle BHD = 60° = \measuredangle BOD \Longrightarrow H \in (OBD).$$

Meanwhile (by design) C, A are the circumcenter and orthocenter of $\triangle HBD$ so lemma 1 means $HA = HC$ or $H \in \ell$. By design $H = X$. □

Lemma 2 – In a triangle with sides a, b, c, circumradius R, circumcenter O and orthocenter H,

$$OH^2 = 9R^2 - (a^2 + b^2 + c^2).$$

Proof. Also omitted. □

Now we extract the answer; in some awful notation, let $BH = u$, $DH = v$. Combine lemma 2 on $\triangle HBD$ with $BD^2 = u^2 - uv + v^2$ (law of cosines) to obtain $AC^2 = (u-v)^2$ and thus the system

$$(u-v)^2 = AC^2 = 78;$$
$$u^2 - uv + v^2 = BD^2 = 169;$$

Homogenizing yields

$$13(u-v)^2 = 6(u^2 - uv + v^2) \implies 7u^2 - 20uv + 7v^2 = 0 \implies \frac{u}{v} = \frac{10 \pm \sqrt{51}}{7} \implies \boxed{068}.$$

Remark. Made in May 2022.