HMMT Team Livesolve/$\TeX$?

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Problem 1

Each person in Cambridge drinks a (possibly different) 12 ounce mixture of water and apple juice, where each drink has a positive amount of both liquids. Marc McGovern, the mayor of Cambridge, drinks $1/6$ of the total amount of water drunk and $1/8$ of the total amount of apple juice drunk. How many people are in Cambridge?

P1

Let total water and juice be w, j respectively; let there be n ppl present.

$$w/6 + j/8 = 12;$$
$$w + j = 12n;$$

Hence $4w + 3j = 288$.

$$\Rightarrow w = 288 - 3(12n) = 288 - 36n;$$

$$\Rightarrow j = 12n - w = 48n - 288;$$

Because $w, j > 0$, $36n < 288 < 48n$, whence $6 < n < 8$ and $n = \boxed{7}$.

Problem 2

2019 students are voting on the distribution of N items. For each item, each student submits a vote on who should receive that item, and the person with the most votes receives the item (in case of a tie, no one gets the item). Suppose that no student votes for the same person twice. Compute the maximum possible number of items one student can receive, over all possible values of N and all possible ways of voting.

P2

Observe that a winner of an item must necessarily receive ≥ 2 votes to win an item. Furthermore, a vote from each person (inc. self) can only be 'used' once. There are 2019 potential votes.
Hence answer $= \lfloor 2019/2 \rfloor = \boxed{1009}$, the end.

Construction?

Obvious: just designate a person to win as much as possible, and pair the others up; have them vote for the designated person in those pairs, at distinct times.

Problem 3

The coefficients of the polynomial $P(x)$ are nonnegative integers, each less than 100. Given that $P(10) = 331633$ and $P(-10) = 273373$, compute $P(1)$.

P3

Definition

Let the **even** and **odd** parts of any function $f(x)$ over $\mathbb{R}$ be $[f(x)+f(-x)]/2$ and $[f(x)-f(-x)]/2$, respectively.

In given prob consider even and odd parts;

Even part: Call Q; $Q(10) = (331633 + 273373)/2 = 302503 = 30 \cdot 10^4 + 25 \cdot 10^2 + 3 \cdot 10^0$;

Odd part: Call R; $R(10) = (331633 - 273373)/2 = 29130 = 29 \cdot 10^3 + 13 \cdot 10^1$.

Hence one may verify that the poly $P(x) = 30x^4 + 29x^3 + 25x^2 + 13x + 3$ works.
$P(1)$ is the sum of the coefficients, which (in this case, at least) is $30 + 29 + 25 + 13 + 3 = 59 + 25 + 16 = \boxed{100}$.

Problem 4

Two players play a game, starting with a pile of N tokens. On each player's turn, they must remove 2^n tokens from the pile for some nonnegative integer n. If a player cannot make a move, they lose. For how many N between 1 and 2019 (inclusive) does the first player have a winning strategy?

P4

Obviously we are assuming optimal play.
Define a **winning config** to be one in which the player who has to move first has a winning strategy. Call all other configs **losing configs**. Below we use engineer's induction. Data in table...

n	1	2	3	4	5	6
V/X	V	V	X	V	V	X

(**V** and **X** are abbreviations for checkmark and X, because laziness ;-;)
Apparent that N is losing config iff $3 \mid N$. Hence, answer is # of ints in $[1, 2019]$ not div. by 3, which is

$$2019 - \left\lfloor \frac{2019}{3} \right\rfloor = \frac{2}{3}(2019) = \boxed{1346}.$$

Problem 5

Compute the sum of all positive real numbers $x \leq 5$ satisfying

$$x = \frac{\lceil x^2 \rceil + \lfloor x \rfloor \lceil x \rceil}{\lfloor x \rfloor + \lceil x \rceil}.$$

P5

So that we may assume $\lceil x \rceil = \lfloor x \rfloor + 1$, we first tackle the trivial case of $x \in \mathbb{Z}$. When $x \in \mathbb{Z} \setminus \{0\}$ we get the trivial equation
$x = (x^2 + x \cdot x)/(x + x), \forall x > 0$
This case gives us a subtotal of $1 + 2 + 3 + 4 + 5 = [15]$.
(It is a personal habit of mine to put subtotals in brackets on scratch during math contests.)

P5 Nontrivial case

Let fractional part of x be $\{x\}$.
Rewrite orig. equation using $\lceil x \rceil = \lfloor x \rfloor + 1$ and $x = \{x\} + \lfloor x \rfloor$...

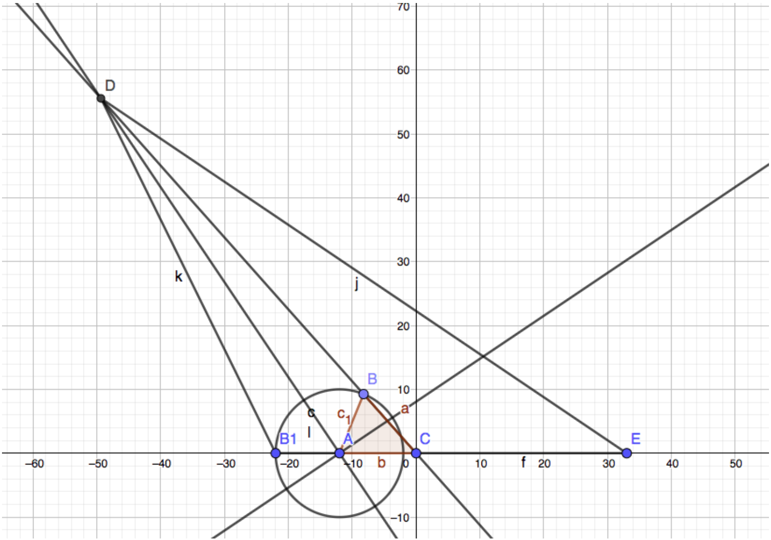
$$\{x\} = \frac{\lceil x^2 \rceil + \lfloor x \rfloor (\lfloor x \rfloor + 1)}{2 \lfloor x \rfloor + 1} - \lfloor x \rfloor = \frac{\lceil x^2 \rceil - \lfloor x \rfloor^2}{2 \lfloor x \rfloor + 1}.$$

Numerator$= 1, 2, \ldots 2 \lfloor x \rfloor + 1$
Fix *floorx*; $2 \lfloor x \rfloor$ possible values, hence sum of int parts is $2 \lfloor x \rfloor^2$.
Frac parts $\to (1 + 2 + \cdots + +2 \lfloor x \rfloor)/(2 \lfloor x \rfloor + 1) = \lfloor x \rfloor$.
Subtotal$= \sum_{k=0}^{4} 2k^2 + k = 3 + 10 + 21 + 36 = [70]$.
Final ans $= 15 + 70 = \boxed{85}$

Problem 8

In $\triangle ABC$, the external angle bisector of $\angle BAC$ intersects line BC at D. E is a point on ray AC such that $\angle BDE = 2\angle ADB$. If $AB = 10, AC = 12$, and $CE = 33$,compute DB.

Ggb'd diag



70
60
50
40
30
20
10
−10
−60
−50
−40
−30
−20
−10
0
10
20
30
40
50
D
k
j
B
c
c_1
a
l
B1
A
C
E
b
f

P8

Solved with *Aditya Chandrasekhar.*

Reflect line CD over line AD, giving us a $B' \in \overline{CA}$. Then $DB/DE = DB'/DE$ by sym.
Also observe that
$\angle CDE = 2\angle CDA = \angle CDA + \angle ADB' = \angle CDB'$, also by sym.
Hence $DB'/DE = CB'/CE$.
Finally, eval'ing CB':

$$CB' = CA + AB' = CA + AB = 22.$$

Hence $DB/DE = DB'/DE = CB'/CE = 22/33 = \boxed{2/3}$.

Post-meeting remarks

Well, what about the other problems?

Too cancerous imo... save your time ig.
Plus no time left smhmh