Gabriel's very long instrument

A demonstration of Beamer for AP Calc BC

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Outline

Quite long indeed. Calculator helpful in computing to come, but not strictly needed.

In a nutshell

Note

To save time on behalf of any potential graders, let it be noted that §2 is solids of revolution (semester 2& no calc), and §3 is related rates (semester 1& calc required). Apart from that the outline is a sufficient synopsis.

Part within Calc BC curriculum

Gabriel's horn is the (infinitely long) solid formed by rotating $y = 1/x$ (for $x \geq 1$) about the x-axis.

I'll begin with the following routine exercise: *what is its volume?*

Volume

Directly computable via method taught in class:

$$V = \pi \int_0^\infty dx/x^2 = \frac{\pi}{x}\bigg|_{x=1}^{\infty} = \pi.$$

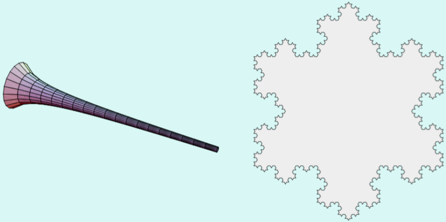
Here, when we write $f(\pm\infty)$, we mean $\lim_{x\to\pm\infty} f(x)$. Slight abuse of notation but not at all fatal.
(If the said limit does not exist, then neither does the original integral.)

Motivation for further investigation

Gabriel's horn is the most typical example of a solid with finite volume but infinite surface area (henceforth *SA* for brevity). This should remind you of the Koch snowflake– finite area, infinite perimeter.

Artwork

Gabriel at left, Koch at right:



The formula

Theorem

Revolve the curve $y = f(x), x \in [a, b]$ about the x-axis. Then the resulting surface has area

$$2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx.$$

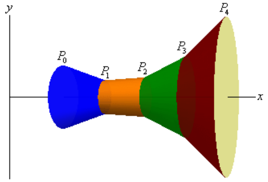
Proof to begin on next slide...

Derivation

Proof can be found on Google, but I'll derive it for the purpose of understanding.

Recall how areas of 2D regions and volumes of solids of revolution were approximated by thin rectangles and circular disks/washers/cylindrical shells.

More relevant is the arclength formula, where curves were approximated by short segments.
For surface area we'll use the lateral areas of frustums, as shown in the diagram below from the internet:



Proof, cont'd.

Putting that picture into words... we'll need the following formula:

Lemma

A frustum with radii r_1, r_2 and slant height ℓ has lateral (non-base) *SA*

$$\pi(r_1 + r_2)\ell.$$

Subproof (or more accurately, part thereof)

We will make full use of 'unrolling'.
WLOG $r_1 < r_2$; when the two radii are equal unrolling gives a rectangle, and when $r_1 > r_2$ we can simply swap the radii. Now we unroll to get a annular sector which is easier to compute.

By similarity/homothety the inner and outer radii are $r_{inner} = r_1\ell/(r_2 - r_1)$ and $r_{outer} = r_2\ell/(r_2 - r_1)$ respectively. Next, the outer circumference of the sector is $2\pi r_2$, so the annular angle (in radians) is

$$\frac{2\pi r_2}{r_{outer}} = \frac{2\pi r_2}{r_2\ell/(r_2 - r_1)} = 2\pi\frac{r_2 - r_1}{\ell}.$$

Finishing the lemma proof

Subproof cont'd.

Now we finish by computing

$$\text{Lateral } SA = \frac{1}{2}(r_{outer}^2 - r_{inner}^2)2\pi\frac{r_2 - r_1}{\ell} = \pi\frac{\left(r_2^2 - r_1^2\right)\ell^2}{(r_2 - r_1)^2}\frac{r_2 - r_1}{\ell}$$

$$= \pi(r_1 + r_2)\ell$$

as desired. □

Finishing the proof

From that, each small frustum with radii at x_0 with width Δ has

$$\text{Lateral } SA = 2\pi \frac{f(x_0) + f(x_0 + \Delta)}{2} \sqrt{(f(x_0 + \Delta) - f(x_0))^2 + \Delta^2}.$$

Now, recall that when Δ gets small, $f(x_0 + \Delta) - f(x_0) \approx \Delta \cdot f'(x_0)$, and $\frac{f(x_0)+f(x_0+\Delta)}{2}$ is the familiar trapezoid-rule area. Thus we may simplify the above as

$$\cdots \approx 2\pi f(x_0)\sqrt{f'(x_0)^2 + 1}\Delta$$

so the exact area is the integral

$$2\pi \int_a^b f(x)\sqrt{f'(x)^2 + 1}dx$$

as desired. ∎

Plugging in formula

SA of the horn

Plugging in $f(x) = 1/x, x \in [1, \infty)$ into the surface area formula... observe that $f'(x) = -1/x^2$. Then

$$SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + 1/x^4}\,dx.$$

Now, we can't directly evaluate this integral by antidifferentiating the integrand, but we can still show that it's nonexistent: bound $\sqrt{1 + 1/x^4} > 1$, so that

$$2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + 1/x^4}\,dx > 2\pi \int_1^\infty \frac{dx}{x},$$

which is easily seen to be nonexistent/divergent.

Remark

Power series could be used for antidifferentiation, but no need...

Maverick remarks

Remark 1

Yes, I know that the AP Calc BC curriculum only includes volume formula(s) for solids of revolution, but I think the surface area formula is important to know as well; there's no reason it's not eligible for the final project, seeing as it's derived from single-variable methods anyway.

Just because an irrelevant organization called the **College Board** renders it useless on its tests doesn't mean we shouldn't learn it as part of single-variable...

Remark 2

OK, time to include some rate exercise to fit the project requirements... I think I've already told the teacher about the surface-area part of this project...

Expanding the horn

OK everyone, time to pull out the (graphing) calc.

Exercise

Suppose that we cut off the horn at some value to make itsquare finite, say $x = 10$. Now, the radius of the horn (y) is uniformly increased at a rate of 1 from the original $y = 1/x$. What is the rate of change of the surface area immediately after the expanding begins?

I claim that this is not computable manually. In the computations to come we will come across a finite integral for which the integrand has no elementary antiderivative.

Remark

It is important to know that the above has infinite exceptions. For instance

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Visualize this as the horn 'blowing up'.

Solution to the exercise

Application of related rates. Formalize by let volume be V, time be t ,so that c is some function of t.
Then $dy/dx \equiv -1/x^2$, leading to

$$SA = 2\pi \int_1^{10} \left(\frac{1}{x} + c\right) \sqrt{1 + (1/x^2)^2} \; dx$$

$$= 2\pi \left[c \int_1^{10} \sqrt{1 + 1/x^4} dx + \int_1^{10} \frac{1}{x} \sqrt{1 + 1/x^4} dx\right].$$

Differentiating the above wrt c gives a rate of

$$\frac{dSA}{dc} = \int_1^{10} \sqrt{1 + 1/x^4} dx$$

which is proportional to the SA (and c)! Let the above integral be C; then we get

$$\left.\frac{dSA}{dt}\right|_{c=0} = \left.\frac{dSA}{dc}\right|_{c=0} \left.\frac{dc}{dt}\right|_{c=0} = C \cdot 1 = C.$$

Finish by computing $C \approx 9.513$ via graphing calc.

Acknowledgements

- This presentation was typeset in LaTeX using the class `beamer` for presentations with TeXLive and `evince`. Handwriting this would be an utter nightmare. . . the erasing and revising would be atrocious;
- Thanks to AoPS user MathJams for their idea contributions- else this project would only consist of the title slide;
- Simpson's rule was used in computing the integrals. Although not taught in AP Calc BC curriculum either, I also would like to advocate its teaching. More accuracy for the same number of subdivisions, how good is that!