Desargues and his meme

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We are lazy, so here is a problem compilation, one of the shortest handouts of all time.

Very brief blurb

Desargues was a sinner, a mastermind spinner,
Mathematical genius, his thoughts got much bigger.
With lines and planes, he played his wicked game,
Geometry was his realm, and he conquered the terrain.

He twisted and turned, in his mathematical maze,
Proving the theorems that left others amazed.
His mind was a canvas, where concepts would collide,
Creating new dimensions, in which truths would reside.

From perspective, he derived his duality,
Projective geometry, his art with clarity.
He saw the world in a different light,
Unveiling hidden symmetries, day and night.

Desargues danced with angels and demons alike,
Challenging the norms, never afraid to strike.
His sins were his passion, his rebellion was clear,
In a world of shapes and numbers, he had no fear.

So raise a toast to Desargues, the sinner with a vision,
Whose mathematical legacy defies all derision.
For in his wickedness, he found the truth,
And left us with a geometric marvel, in our youth.
– ChatGPT '23

0 Acknowledgement

Eric Shen for teaching me this black magic and its very interesting applications. Thanks so much Eric! – Neal

1 Opening examples

Example 1 (OMMC Main 2023/24 by Tiger)

Define acute $\triangle ABC$ with circumcenter O. The circumcircle of $\triangle ABO$ meets segment BC at $D \neq B$, segment AC at $F \neq A$, and the Euler line of $\triangle ABC$ at $P \neq O$. The circumcircle of $\triangle ACO$ meets segment BC at $E \neq C$. Let $\overline{BC}$ and $\overline{FP}$ intersect at X, with C between B and X. If $BD = 13$, $EC = 8$, and $CX = 27$, find DE.

2 Problems

Approximately increasing difficulty....

Problem 1 (USA TST 2004/4). Let ABC be a triangle. Choose a point D in its interior. Let ω_1 be a circle passing through B and D and ω_2 be a circle passing through C and D so that the other point of intersection of the two circles lies on AD. Let ω_1 and ω_2 intersect side BC at E and F, respectively. Denote by X the intersection of DF, AB and Y the intersection of DE, AC. Show that $XY \parallel BC$.

Problem 2 (CJMO 2021/1). Let ABC be an acute triangle, and let the feet of the altitudes from A, B, C to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ be D, E, F, respectively. Points $X \neq F$ and $Y \neq E$ lie on lines CF and BE respectively such that $\angle XAD = \angle DAB$ and $\angle YAD = \angle DAC$. Prove that X, D, Y are collinear.

Problem 3 (IGO 2018/I5). Suppose that $ABCD$ is a parallelogram such that $\angle DAC = 90^o$. Let H be the foot of perpendicular from A to DC, also let P be a point along the line AC such that the line PD is tangent to the circumcircle of the triangle ABD. Prove that $\angle PBA = \angle DBH$.

Problem 4 (Serbia 2017/6). Let k be the circumcircle of $\triangle ABC$ and let k_a be A-excircle .Let the two common tangents of k, k_a cut BC in P, Q.Prove that $\angle PAB = \angle CAQ$.

Problem 5 (Taiwan TST 2014/3/3). Let ABC be a triangle with circumcircle Γ and let M be an arbitrary point on Γ. Suppose the tangents from M to the incircle of $\triangle ABC$ intersect $\overline{BC}$ at two distinct points X_1 and X_2. Prove that the circumcircle of triangle MX_1X_2 passes through the tangency point of the A-mixtilinear incircle with Γ.

Problem 6 (USA TST 2018/5 (by Evan)). Let $ABCD$ be a convex cyclic quadrilateral which is not a kite, but whose diagonals are perpendicular and meet at H. Denote by M and N the midpoints of $\overline{BC}$ and $\overline{CD}$. Rays MH and NH meet $\overline{AD}$ and $\overline{AB}$ at S and T, respectively. Prove that there exists a point E, lying outside quadrilateral $ABCD$, such that

- ray EH bisects both angles $\angle BES$, $\angle TED$, and
- $\angle BEN = \angle MED$.

Problem 7 (IMO 2019/2). In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

Problem 8 (MOP HW #21). In acute scalene $\triangle ABC$ with circumcenter O, orthocenter H, Kosnita point $X_{54} = K$, define $P = (HO) \cap (BOC)$, Q be the foot from line onto AO. Prove that P, Q, K are collinear. (The Kosnita point is the point at which the line through A and the circumcenter of $\triangle BOC$ and the other two analogous lines concur; it is the isogonal conjugate of the nine-point center.

Problem 9 (Shortlist 2012/G8). Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Problem 10 (Shortlist 2021/G8). Let ABC be a triangle with circumcircle ω and let Ω_A be the A-excircle. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $\overline{AR} \perp \overline{BC}$.

2.1 Addendum

Problem 11 (USAMO 2012/5). Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A', B', C' are collinear.

Problem 12 (Shortlist 2022/G8). Let $AA'BCC'B'$ be a convex cyclic hexagon such that AC is tangent to the incircle of the triangle $A'B'C'$, and $A'C'$ is tangent to the incircle of the triangle ABC. Let the lines AB and $A'B'$ meet at X and let the lines BC and $B'C'$ meet at Y.

Prove that if $XBYB'$ is a convex quadrilateral, then it has an incircle.

Problem 13 (China 2020/2). Let ABC be a triangle, and let the bisector of $\angle A$ intersect $\overline{BC}$ at D. Point P lies on line AD such that P, A, D are collinear in that order. Suppose $\overline{PQ}$ is tangent to (ABD) at Q, $\overline{PR}$ is tangent to (ACD) at R, and Q and R lie on opposite sides of line AD. Let $K = BR \cap CQ$. Prove that if the line through K parallel to BC intersects lines QD, AD, RD at E, L, F, respectively, then $EL = KF$.

2.1.1 Are these (D)DIT?

I have not done them, but there are apparently (D)DIT solutions to the below problems.

Problem 14 (Shortlist 2022/G3). Let $ABCD$ be a cyclic quadrilateral. Assume that the points Q, A, B, P are collinear in this order, in such a way that the line AC is tangent to the circle ADQ, and the line BD is tangent to the circle BCP. Let M and N be the midpoints of segments BC and AD, respectively. Prove that the following three lines are concurrent: line CD, the tangent of circle ANQ at point A, and the tangent to circle BMP at point B.

Problem 15 (TSTST 2023/6). Let ABC be a scalene triangle and let P and Q be two distinct points in its interior. Suppose that the angle bisectors of $\angle PAQ, \angle PBQ$, and $\angle PCQ$ are the altitudes of triangle ABC. Prove that the midpoint of $\overline{PQ}$ lies on the Euler line of ABC. (the author was splashed again :skull:)